Lecture 24

Tuesday, April 6, 2021 3:36 PM

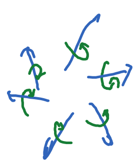
* Prayer
* Spiritual thought : general conference
* Answering questions....

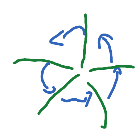
Properties of divergence and curl :

$\operatorname{div}(\operatorname{curl}) = 0$

$\operatorname{curl}(\operatorname{grad}) = 0$ $\rightsquigarrow$ a method to show if a vector field is conservative

$\left\{ \begin{array}{l} \text{curl} = 0 \\ \text{domain is simply conn.} \end{array} \right. \Rightarrow$ cons.





what if $\operatorname{div}(\operatorname{curl}) \neq 0$?

What if $\operatorname{curl}(\operatorname{grad}) \neq 0$

Ex $F = \langle x, y, z \rangle$

Is F conservative ?

$G = \langle y, z, x \rangle$

Is G conservative ?

Ex $F = \langle x, y, z \rangle$

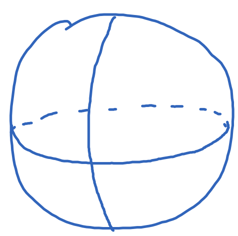
Is F the curl of any vector field ?

Surfaces

$(u,v) \in D$ $\longrightarrow$ $r(u,v) \in S$

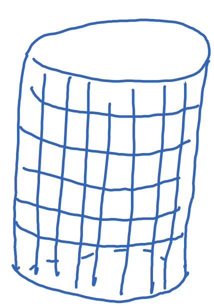
Ex: 1) The sphere $x^2+y^2+z^2=1$

$$r: \begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases} \qquad \begin{array}{l} 0 \le \phi \le \pi \\ 0 \le \theta \le 2\pi \end{array}$$



2) The cylinder $x^2+y^2=1$, $0 \le z \le 1$

$$r: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le 1 \end{array}$$



3) The Mobius strip:

$$\begin{cases} x = 2\cos\theta + r\cos\frac{\theta}{2} \\ y = 2\sin\theta + r\cos\frac{\theta}{2} \\ z = r\sin\frac{\theta}{2} \end{cases}$$

$0 \le \theta \le 2\pi$

$\frac{1}{2} \le r \le \frac{1}{2}$

4) The torus:

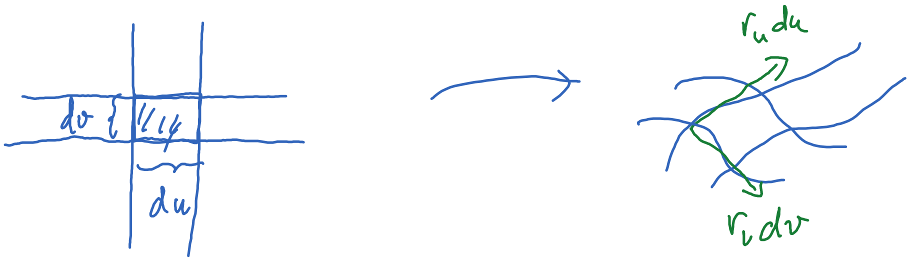
$$\begin{cases} x = (a + r\cos\phi)\cos\theta \\ y = (a + r\cos\phi)\sin\theta \\ z = r\sin\phi \end{cases}$$

$0 \le \theta \le 2\pi$

$0 \le \phi \le 2\pi$

(one can take $a = 2$, $r = 1$)

Area of a surface



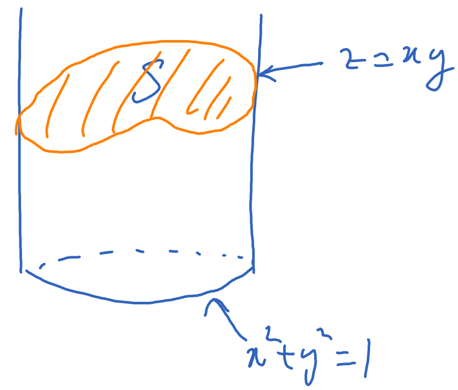
old area = $du\,dv$

new area = $|r_u du \times r_v dv|$

$= |r_u \times r_v|\,du\,dv$

Area of surface $S = \iint_D |r_u \times r_v|\,dA$

Ex

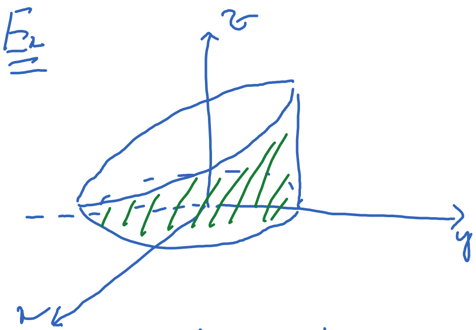


$$r: \begin{cases} x = x \\ y = y \\ z = xy \end{cases} \qquad (x,y)\in D \leftarrow \text{unit disc}$$

$$r_x = \langle 1, 0, y\rangle, \quad r_y = \langle 0, 1, x\rangle$$

$$\text{Area} = \iint_D \sqrt{x^2+y^2+1}\, dA = \int_0^1\int_0^{2\pi} \sqrt{r^2+1}\, r\, d\theta\, dr = \cdots$$

Ex



$$r: \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases} \qquad \underbrace{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le \sin\theta + 1 \end{array}}_{D}$$

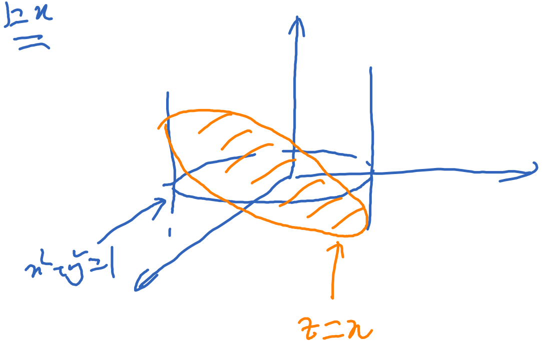
$x^2+y^2=1$

cut by the planes $z = y+1$ and $z=0$

$$\text{Area} = \iint_D |r_\theta \times r_z|\, dA = \iint_D \underbrace{|\langle -\sin\theta, \cos\theta, 0\rangle \times \langle 0,0,1\rangle|}_{\underbrace{|\langle \cos\theta, \sin\theta, 0\rangle|}_{1}}\, dA = \text{area}(D)$$

$$\begin{matrix} -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{matrix}$$

Ex



Two ways:

$$R_1: \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r\cos\theta \end{cases} \qquad \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$R_2: \begin{cases} x = x \\ y = y \\ z = x \end{cases} \qquad (x,y) \in \underset{\text{disc}}{D}$$

$$\text{Area} = \int_0^{2\pi}\int_0^1 \left|\frac{\partial R_1}{\partial r} \times \frac{\partial R_1}{\partial \theta}\right| dr\,d\theta$$

$$= \iint_D \left|\frac{\partial R_2}{\partial x} \times \frac{\partial R_2}{\partial y}\right| dA = \int_0^{2\pi}\int_0^1 \left|\frac{\partial R_2}{\partial x} \times \frac{\partial R_2}{\partial y}\right| r\,dr\,d\theta$$

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* Integral over a surface

Scalar: $\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(r(u,v))\,\underbrace{|r_u \times r_v|\,dA}_{\text{area of a patch}}$

Ex $\displaystyle\iint_S xyz\,dS$

$$S: \begin{cases} x = u\cos v \\ y = u\sin v \\ z = u \end{cases} \qquad \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le \frac{\pi}{2} \end{matrix}$$